Anisotropic Homogeneous Cosmology in the Nonsymmetric Theory of Gravitation

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Abstract

Solutions of the field equations of the Nonsymmetric Gravitational Theory with $g_{[0i]} = 0$ are obtained for the homogeneous, plane-symmetric, time-dependent case, both in vacuum and in the presence of a perfect fluid. Cosmological consequences include a dependence of the speed of light on its polarisation, as in a birefringent crystal.

1. Introduction

As an alternative to Einstein's theory of gravitation (GR), the Nonsymmetric Theory of Gravitation (NGT) possesses a richer structure with the potential for significantly different physical predictions. This structure, first proposed by Einstein and Straus (1945, 1946) in an attempt to unify gravitation and electromagnetism, is characterised by a nonsymmetric fundamental tensor $\boldsymbol{g}$. The tangent bundle of space-time is endowed with a hypercomplex algebraic structure (Crumeyrolle 1967, Kunstatter *et al* 1983, Mann 1984, 1989, Moffat 1984) compatible with $\boldsymbol{g}$ and with the connections. In terms of components, one has $g_{\mu\nu} = g_{(\mu\nu)} + e\, g_{[\mu\nu]}$, where $e^2 = +1$, and round and square brackets stand for symmetrisation and antisymmetrisation, respectively.

Although this attempt at unification was not successful, Moffat (1979) suggested that the extended structure could be interpreted as a generalised theory of gravitation. For several years thereafter, the physical content of the theory was thought (Moffat 1991) to arise from the existence of a new conserved charge which entered in the time-space components ($g_{[i0]}$) of $\boldsymbol{g}$, as well as in its diagonal ones (line element), but with the space-space components ($g_{[ij]}$) being put equal to zero. After it was pointed out (Damour *et al* 1992, 1993) that with this choice, the theory, unprotected by gauge invariance, would allow coupled negative-energy modes to propagate to infinity, a version with $g_{[ij]} \neq 0$ was proposed (Cornish and Moffat 1994, Légaré and Moffat 1995, Moffat 1995a, 1995b, 1995c) which avoids this defect, as well as other stability problems (Clayton 1996, 1997), as long as the $g_{[i0]}$ can be put equal to zero. Claims (Moffat and Sokolov 1995, Moffat 1995c) and counterclaims (Burko and Ori 1995) have been made since then about the absence of apparent event horizons under dynamical conditions in the spherically-symmetric case. Unfortunately, only the static vacuum spherically-symmetric solution has been obtained analytically (Wyman 1950) for $g_{[i0]} = 0$, and it is difficult to infer from it the behaviour of time-dependent solutions.

This paper presents the first such analytic dynamical solution, but instead of the spherically-symmetric case, we address the more tractable homogeneous plane-symmetric case which may have applications to cosmology. Section 2 sets the relevant field equations to be solved. Very little theoretical background is provided as this is easily accessible in the existing literature (see for instance Moffat 1995b). Section 3 presents the solution to the vacuum case, and Section 4 the solution to the case with matter. Section 5 contains a few comments about the matter solution, in particular expressions containing NGT corrections for the GR prediction for the rate of expansion of the universe. In Section 6 we derive one major consequence of the direct coupling of NGT to electromagnetic fields: the universe can behave somewhat like a birefringent crystal so that the polarisation vector of light is rotated as it travels over cosmic distances. The paper ends with a few concluding remarks in Section 7.

2. Field Equations

In this paper, boldface symbols denote tensors, while a tilde above a tensor makes it a density. We work in units where $G = c = 1$. Finally, $\dot{X} \equiv X_{,t} \equiv \partial X/\partial t$.

In the homogeneous, plane-symmetric case, we use the fundamental tensor:

$$\boldsymbol{g} = e^{\nu} dt \otimes dt - e^{\lambda} dx \otimes dx - \beta(dy \otimes dy + dz \otimes dz) + f(dy \wedge dz), \tag{2.1}$$

where, without loss of generality, the yz plane has been chosen as the plane of symmetry, and ν, λ, β and f are functions of time.

Denoting by $g^{\mu\nu}$ the contravariant components of $\boldsymbol{g}$, the *homogeneous, plane-symmetric* connection components obey

$$\tilde{g}^{\mu\nu}{}_{,\lambda} + \tilde{g}^{\rho\nu}\Gamma^{\mu}_{\rho\lambda} + \tilde{g}^{\mu\rho}\Gamma^{\nu}_{\lambda\rho} - \tilde{g}^{\mu\nu}\Gamma^{\rho}_{(\lambda\rho)} = 0 \tag{2.2}$$

Define the Einstein tensor, $\boldsymbol{G} \equiv \boldsymbol{R} - \frac{1}{2}\boldsymbol{g}\,\mathrm{Tr}\boldsymbol{R}$, where the covariant components of $\boldsymbol{R}$ are given by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \tfrac{1}{2}(\Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\nu\alpha,\mu}) - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \tag{2.3}$$

Then, the other homogeneous, plane-symmetric field equations reduce to

$$\boldsymbol{G} \;=\; 8\pi\boldsymbol{T} \tag{2.4}$$

where a possible contribution from a cosmological constant has been ignored.

We assume matter to consist of a perfect fluid made of particles moving with four-velocity $\boldsymbol{u}$. The energy-momentum tensor takes the form (Vincent 1985):

$$\boldsymbol{T} \;= (\rho + p)\boldsymbol{u}\otimes\boldsymbol{u} - p\boldsymbol{g}, \tag{2.5}$$

with ρ the internal energy density and p the pressure. We choose a comoving frame, where $u^i = 0$ $(i = x, y, z)$, and $u^t = 1$. Since only three of the four equations (2.4) will be independent, one of the four functions in (2.1) is arbitrary; it makes sense to take $\nu = 0$. The time t is interpreted as the proper time of the particles in the fluid.

Now, introduce the parametrisation $\beta = R^2\cos\psi$, and $f = R^2\sin\psi$, as well as the definition $A = \log R^2$. The nonvanishing connection components which solve (2.2) read

$$\begin{aligned}
\Gamma^{t}_{xx} &= \tfrac{1}{2}\dot\lambda e^{\lambda}, & \Gamma^{t}_{yy} &= \Gamma^{t}_{zz} = \tfrac{1}{2}(\beta\dot A - f\dot\psi)\\
\Gamma^{x}_{(tx)} &= \tfrac{1}{2}\dot\lambda, & \Gamma^{t}_{[yz]} &= -\tfrac{1}{2}(f\dot A + \beta\dot\psi)\\
\Gamma^{z}_{(tz)} &= \tfrac{1}{2}\dot A, & \Gamma^{y}_{[tz]} &= -\Gamma^{z}_{[ty]} = \tfrac{1}{2}\dot\psi
\end{aligned} \tag{2.6}$$

With the help of (2.3) and (2.6), (2.4) can now be written down in terms of the components of $\boldsymbol{g}$. Indeed, the nonzero mixed components of $\boldsymbol{G}$ are

$$G_t{}^t \;= \tfrac{1}{4}(\dot A^2 - \dot\psi^2) + \tfrac{1}{2}\dot A\dot\lambda = 8\pi\rho \tag{2.7}$$

$$G_x{}^x \;= \ddot A + \tfrac{1}{4}(3\dot A^2 + \dot\psi^2) = -8\pi p \tag{2.8}$$

$$\begin{aligned}
G_y{}^y &= G_z{}^z\\
&= \tfrac{1}{2}\ddot A + \tfrac{1}{4}(\dot A^2 + \dot\psi^2) + \tfrac{1}{4}\dot A\dot\lambda + \tfrac{1}{4}\dot\lambda^2 + \tfrac{1}{2}\ddot\lambda\\
&= -8\pi p
\end{aligned} \tag{2.9}$$

$$G_y{}^z \;=\; \ddot\psi + \dot A\dot\psi + \tfrac{1}{2}\dot\lambda\dot\psi = 0 \tag{2.10}$$

The matter-response equation for the perfect-fluid $\boldsymbol{T}$, (2.5), is

$$\dot\rho \;=\; -(\rho + p)(\dot A + \tfrac{1}{2}\dot\lambda) \tag{2.11}$$

3. Vacuum Solution

With $\rho = p = 0$, it is convenient to form the combination $\frac{1}{2}(G_t{}^t + G_x{}^x + 2G_y{}^y)$, *i.e.*, $\ddot{\lambda} + \dot{\lambda}(\dot{A} + \frac{1}{2}\dot{\lambda}) = 0$, which is immediately integrated:

$$(e^{\lambda/2})_{,t} = c/R^2 \tag{3.1}$$

Likewise, the combination $\frac{1}{2}(G_t{}^t + G_x{}^x)$, *i.e.*, $\ddot{A} + \dot{A}^2 + \frac{1}{2}\dot{A}\dot{\lambda} = 0$, gives

$$\dot{A} = \frac{c_1}{R^2 e^{\lambda/2}} \tag{3.2}$$

Combining (3.1) and (3.2) leads to a second-order equation for R with solution:

$$R^2 = R_0^2 (t/t_0)^{\frac{c_1}{c+c_1}} \tag{3.3}$$

Then

$$e^{\lambda} = (c + c_1)^2 (t/t_0)^{\frac{2c}{c+c_1}} \tag{3.4}$$

Now, integrating (2.10) and combining the result with (3.3) and (3.4) yields

$$\psi = \psi_0 + \frac{c_0}{c+c_1} \log(t/t_0) \tag{3.5}$$

In these equations, c, c_0, and c_1 are constants of integration which are not all independent. Indeed (2.7) and (2.8) (with $\rho = p = 0$) are satisfied only if $\kappa^2 \equiv (c_0/c_1)^2 = 1 + 4c/c_1$. From this, the final form for the vacuum solution becomes:

$$\begin{aligned} e^{\lambda-\lambda_0} &= (t/t_0)^{\frac{-2(1-\kappa^2)}{3+\kappa^2}} \\ R/R_0 &= (t/t_0)^{\frac{2}{3+\kappa^2}} \\ \psi &= \psi_0 + \frac{4\kappa}{3+\kappa^2} \log(t/t_0) \end{aligned} \tag{3.6}$$

Alternatively, a compact form for β and f is written in terms of the complex parameter $e^q \equiv \beta + \mathrm{i}f = R^2 e^{\mathrm{i}\psi}$:

$$e^{q-q_0} = (t/t_0)^{\frac{4(1+\mathrm{i}\kappa)}{3+\kappa^2}} \tag{3.7}$$

As in Einstein's General Relativity, whose corresponding solution (Kasner 1921) can be recovered by setting $\kappa = 0$ and $\psi_0 = n\pi$, $n = 0, 2, 4...$, the solution is open in the plane of symmetry and closed in the perpendicular direction if $\kappa^2 < 1$. If $\kappa^2 > 1$, however, the solution is open in all three spatial directions.

Assuming that the metric is given by $g_{(\mu\nu)}$, its signature changes sign every time $\cos\psi = 0$ since $\beta = R^2 \cos\psi$. This occurs an infinite number of times over a finite interval starting at $t = 0$, although the interval can be made as small as desired by decreasing the NGT parameter κ.

At $t = 0$, both R and the 4-volume element $\sqrt{-g}$ go to zero, whereas g_{11} ($-e^{\lambda}$) goes to infinity if $\kappa^2 < 1$ and to zero if $\kappa^2 > 1$. The solution would thus appear to contain an essential singularity. This is supported by a computer calculation of the Kretschmann scalar, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, with a symbolic manipulator; the result, which we do not reproduce here, contains an additive term that goes like $1/t^4$, along with other terms, also in $1/t^4$, but whose numerator oscillates infinitely fast at $t = 0$.

4. Solution with Matter

In this case, it is best to start from a combination which contains neither ρ nor p. If we subtract (2.9) from (2.8), we obtain after some rearrangement an equation which is also free of ψ:

$$(\log\, R^2 e^{-\lambda})_{,tt} + (\log\, R^2 e^{\lambda/2})_{,t} (\log\, R^2 e^{-\lambda})_{,t} = 0 \tag{4.1}$$

At this point, it is convenient to change the independent variable t to

$$v \equiv \int \frac{dt}{R^2 e^{-\lambda/2}} \tag{4.2}$$

Then we obtain from (4.1)

$$(\log\, R^2 e^{-\lambda})_{,vv} = 0 \tag{4.3}$$

whose solution reads

$$R^2 = e^{\lambda + \alpha(v - v_0)} \tag{4.4}$$

whereas (2.10) becomes $\psi_{,vv} = 0$ and has for solution:

$$\psi = \sigma(v - v_0) \tag{4.5}$$

In (4.4) and (4.5), α and σ are arbitrary constants.

Now we assume an adiabatic equation of state for the matter: $p = \gamma\rho$, with $0 \leq \gamma \leq 1$. From (2.7) and (2.8), the combination $\frac{1}{2}({G_t}^t + {G_x}^x)$, expressed in terms of v, is

$$A_{,vv} = 8\pi(1-\gamma) R^4 e^{\lambda} \rho \tag{4.6}$$

With all these results, we can write (2.7) in terms of v and of a new dimensionless dependent variable x:

$$x_{,v} = \sqrt{6\pi b}(1-\gamma)\left(x^2 - \eta^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4.7}$$

where

$$x^2 \equiv R^4 e^{\lambda} \rho / b + \eta^2 \tag{4.8}$$

and $\eta^2 \equiv (\sigma^2 + \alpha^2/3)/32\pi b$, with b an arbitrary constant. The integral of (4.7) is

$$x = -\eta\, \coth[\sqrt{6\pi b}\,\eta(1-\gamma)(v - v_0)] \tag{4.9}$$

Now, from the conservation equation (2.11), $\rho(R^2 e^{\lambda/2})^{1+\gamma}$ is equal to a constant which we identify with b, and this allows us to write (4.8) as $x^2 - \eta^2 = (R^2 e^{\lambda/2})^{1-\gamma}$.

Then, with the help of (4.4), we can eliminate $\exp[3\alpha(1-\gamma)v/2]$ between (4.8) and (4.9) to find an expression for R as a function of x:

$$R = \left[(x+\eta)^{1-\alpha/\sqrt{\alpha^2+3\sigma^2}}(x-\eta)^{1+\alpha/\sqrt{\alpha^2+3\sigma^2}}\right]^{\frac{1}{3(1-\gamma)}} \tag{4.10}$$

Also, by virtue of (4.9), (4.5) becomes

$$\psi = \frac{4}{\sqrt{3}} \frac{\sigma}{\sqrt{\alpha^2/3 + \sigma^2}} \frac{1}{\gamma - 1} \text{arctanh}\left(\frac{\eta}{x}\right) \tag{4.11}$$

Finally, from (4.4), there comes

$$e^{\lambda} = \left[(x+\eta)^{1-2\alpha/\sqrt{\alpha^2+3\sigma^2}} (x-\eta)^{1+2\alpha/\sqrt{\alpha^2+3\sigma^2}} \right]^{\frac{2}{3(1-\gamma)}} \tag{4.12}$$

To find x as a function of time, we differentiate (4.8) with respect to time and note that $\dot{v} = (R^2 e^{\lambda/2})^{-1}$. Then

$$\sqrt{6\pi b}(1-\gamma)(t-t_0) = \int (x^2-\eta^2)^{\frac{\gamma}{1-\gamma}}\, dx \tag{4.13}$$

Equations (4.10)–(4.13) constitute the general time-dependent plane-symmetric NGT solution for an adiabatic equation of state and $g_{[0i]} = 0$ It reduces to the solution obtained in General Relativity (Schücking and Heckmann 1958) in the limit $\sigma \to 0$. From now on, however, we put equal to zero the constant α characterising the anisotropy due to matter in (4.4); any remaining anisotropy is due to NGT. In that case we are left with

$$\begin{aligned} R &= \left(x^2-\eta^2\right)^{\frac{1}{3(1-\gamma)}} \\ \psi &= \frac{4}{\sqrt{3}} \frac{1}{\gamma-1} \operatorname{arctanh}\left(\frac{\eta}{x}\right) \\ e^{\lambda} &= R^2 \\ \rho &= \rho_0 (R/R_0)^{-3(1+\gamma)} \end{aligned} \tag{4.14}$$

where now, $\eta = \sigma/\sqrt{32\pi b}$, a dimensionless constant independent of initial conditions. We can also write the first two equations of (4.14) as

$$\beta + \mathrm{i} f = \left[(x+\eta)^{1-\mathrm{i}\sqrt{3}} (x-\eta)^{1+\mathrm{i}\sqrt{3}} \right]^{\frac{2}{3(1-\gamma)}} \tag{4.15}$$

The dependence of x on time is to be found from (4.13). For instance, if $\gamma = 0$ (matter-dominated universe), $6\pi b\, t^2 = x^2$, so that

$$R = R_0 (6\pi\rho_0 t^2 - \eta^2/R_0^3)^{1/3} \tag{4.16}$$

whereas, if $\gamma = 1/3$ (radiation-dominated universe), $R^2 = (x^2-\eta^2)$, with x given by the implicit relation

$$\sqrt{6\pi\rho_0}\, t = \frac{3}{4R_0} \left[x(x^2-\eta^2)^{1/2} - \eta^2 \log\left(x + (x^2-\eta^2)^{1/2} \right) \right] \tag{4.17}$$

It is clear from (4.14) and (4.15) that this solution becomes isotropic in the large x (large time) limit, in the sense that $f \to 0$ and $\beta \to e^{\lambda}$, so that we recover the Friedmann-Robertson-Walker solution of General Relativity. It is equally clear that at some time $t_{\mathrm{S}} < t_0$, there occurs a branch-point at $x = \eta$; R and e^{λ} go to zero, and the matter density becomes infinite. When $\gamma = 0$,

$$\begin{aligned} t_{\mathrm{S}} &= \eta/\sqrt{6\pi b} \\ &= \frac{\sigma/\sqrt{3}}{8\pi b} \end{aligned} \tag{4.18}$$

One can show from (4.13) that, both for a dust and a radiation universe, the interval between t_0 and t_{S} is smaller in the NGT solution than in the corresponding one in GR.

5. Early-time Behaviour and Correction to the Age of the Universe

At $t = t_0$, the angle ψ takes the value

$$\psi_0 = \frac{4}{\sqrt{3}} \frac{1}{\gamma - 1} \operatorname{arctanh} \sqrt{\frac{\eta^2}{R_0^3 + \eta^2}} \tag{5.1}$$

whereas $\psi(t_S) \to -\infty$. Thus, β and f must change signs an infinite number of times between t_S and t_0. The zeros of β occur when $\cos \psi = 0$, at

$$x_n = \eta \coth \left[\frac{\sqrt{3}}{8} \pi (1 - \gamma)(2n - 1) \right] \qquad (n = 1, 2, \ldots) \tag{5.2}$$

For a dust-filled universe, the first zero of β is at $x_1 = 1.7\eta \approx \sigma/6b$. All other zeros lie between η and 1.7η. The time elapsed between t_1 and t_0 is

$$t_0 - t_1 = \frac{1}{\sqrt{6\pi\rho_0}} \left[\left(1 + \frac{\eta^2}{R_0^3} \right)^{1/2} - 1.7 \frac{\eta}{R_0^{3/2}} \right] \tag{5.3}$$

The behaviour of β over the interval $0 < t < t_1$ has no obvious physical meaning, but it can be shrunk to unobservable size by reducing the value of η (or σ). Then, if we take t_0 as the present time, expression (5.3) can be interpreted as a rough estimate of the age of the universe.

Now, another way of expressing the age of the universe is with the Hubble parameter, here defined as the average rate of expansion of its 3-volume: $H = \frac{1}{3}(\log \beta e^{\lambda/2})_{,t}$, or, if we use the (R, ψ) representation, $H = \dot{R}/R - \frac{1}{3}\dot{\psi} \tan \psi$. Also, (2.8) can be cast in the form

$$\left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3}\rho + \frac{1}{12}\dot{\psi}^2 \tag{5.4}$$

Insertion of the solution with matter then yields, to first order in $(\sigma/R_0^3)^2$:

$$\begin{aligned} H^2 &= \frac{8\pi}{3}\rho + \frac{1}{12} \left(\frac{\sigma}{R_0^3} \right)^2 \left(\frac{\rho}{\rho_0} \right)^{\frac{2}{1+\gamma}} \left(1 + \frac{16}{3(1-\gamma)} \right) \\ &= \frac{8\pi}{3} \left(\frac{R}{R_0} \right)^{-3(1-\gamma)} + \frac{1}{12} \left(\frac{\sigma}{R_0^3} \right)^2 \left(\frac{R}{R_0} \right)^{-6} \left(1 + \frac{16}{3(1-\gamma)} \right) \end{aligned} \tag{5.5}$$

From this comes a relation, valid at the present time, between the Hubble parameter, the matter density, and σ/R_0^3:

$$H_0^2 = \frac{8\pi}{3}\rho_0 + \frac{19}{36} \left(\frac{\sigma}{R_0^3} \right)^2 \tag{5.6}$$

Now we can express our "age of the universe" in (5.3) as

$$t_0 - t_1 \approx \frac{2}{3H_0} \left(1 - \frac{1}{15} \frac{\sigma}{b} \right) \tag{5.7}$$

This last form is convenient because both σ and $b = \rho R^3$ remain constant throughout the time evolution. But no very stringent bound can be inferred from it. For instance, if $\sigma/b = 0.1$, the length of the matter-dominated era is shortened by one percent only.

In a radiation-dominated universe ($\gamma = 1/3$), we have instead

$$H^2 = \frac{8\pi}{3}\rho + \frac{3}{4}\left(\frac{\sigma}{R_0^3}\right)^2\left(\frac{\rho}{\rho_0}\right)^{3/2} \tag{5.8}$$

or, in terms of black-body radiation temperature,

$$H^2 = \frac{8\pi a}{3}T^4\left[1 + \frac{9}{32\pi a}\left(\frac{\sigma}{T_0^2 R_0^3}\right)^2\left(\frac{T}{T_0}\right)^2\right] \tag{5.9}$$

where a is the Stefan constant, and σ is the same as in the dust solution so as to ensure proper matching of the two cases.

6. Birefringent Universe

As a nonmetric theory of gravitation, NGT couples matter fields directly to $g_{[\mu\nu]}$ in a way that generically violates the Einstein equivalence principle. For instance, the propagation of electromagnetic waves should be affected by the coupling between $g_{[\mu\nu]}$ and electromagnetic fields. This was first pointed out by Mann and Moffat (1981) and detailed calculations were carried out (Will 1989, Gabriel *et al* 1990, 1991a, 1991b, 1991c) for $g_{[i0]} \neq 0$, $g_{[ij]} = 0$ in the static spherically-symmetric case. In the cosmological plane-symmetric solution with matter discussed here, we should also find some dependence of the propagation of light on its polarisation.

The coupling of NGT to electromagnetism is written as an electromagnetic Lagrangian density which has the general form (Mann *et al* 1989)

$$\mathcal{L}_{\rm em} = -\frac{1}{16\pi}\sqrt{-g}\mathcal{F}g^{\mu\alpha}g^{\nu\beta}[ZF_{\mu\nu}F_{\alpha\beta} + (1-Z)F_{\alpha\nu}F_{\mu\beta} + YF_{\mu\alpha}F_{\nu\beta}] \tag{6.1}$$

where Y and Z are constants, and $\mathcal{F} = 1$ when $g_{[\mu\nu]} = 0$. As usual, the electromagnetic field tensor, $F_{\mu\nu}$, can be decomposed into electric components, $F_{i0} = E_i$, and magnetic components, $F_{ij} = \epsilon_{ijk}B^k$. We also take $\mathcal{F} = \sqrt{-g}/\sqrt{-\det\, g_{(\mu\nu)}}$. With (2.1) and $R^2 = e^\lambda$ in (4.14), (6.1) can then be written

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}\left[\epsilon_\parallel \boldsymbol{E}_\parallel^2 + \epsilon_\perp \boldsymbol{E}_\perp^2 - \left(\frac{\boldsymbol{B}_\parallel^2}{\mu_\parallel} + \frac{\boldsymbol{B}_\perp^2}{\mu_\perp}\right)\right] \\
&= \frac{1}{8\pi}\left[\epsilon_\parallel\left(\boldsymbol{E}^2 + (\frac{\epsilon_\perp}{\epsilon_\parallel} - 1)(\boldsymbol{n}\cdot\boldsymbol{E})^2\right) - \frac{1}{\mu_\parallel}\left(\boldsymbol{B}^2 - \left(1 - \frac{\mu_\parallel}{\mu_\perp}\right)(\boldsymbol{n}\cdot\boldsymbol{B})^2\right)\right]
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
\epsilon_\parallel &= \mu_\parallel = R \\
\epsilon_\perp &\equiv \epsilon_\parallel / \cos\psi \\
\mu_\perp &= \mu_\parallel \cos\psi (1 - 2X\sin^2\psi)^{-1}
\end{aligned} \tag{6.3}$$

and $\boldsymbol{n}$ is a unit vector perpendicular to the plane of symmetry. One may think of $\boldsymbol{\epsilon}$ as a diagonal tensor with nonzero components $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}) = (\epsilon_\perp, \epsilon_\parallel, \epsilon_\parallel)$, and similarly for a diagonal tensor $\boldsymbol{\mu}$. Our choice of $\mathcal{F}$ ensures that $\epsilon_\parallel = \mu_\parallel$, so that any NGT effect must arise from the $(\boldsymbol{n}\cdot\boldsymbol{E})^2$ and $(\boldsymbol{n}\cdot\boldsymbol{B})^2$ terms in $\mathcal{L}$.

The Maxwell-NGT equations in momentum space which follow from varying (6.2) are

$$\begin{aligned} \boldsymbol{k}\cdot\boldsymbol{D} &= 0 \\ \boldsymbol{k}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{k}\wedge\boldsymbol{E}-\omega\boldsymbol{B} &= 0 \\ \boldsymbol{k}\wedge\boldsymbol{H}+\omega\boldsymbol{D} &= 0 \end{aligned} \tag{6.4}$$

where we have defined $\boldsymbol{D}\equiv\boldsymbol{\epsilon}\cdot\boldsymbol{E}$, and $\boldsymbol{B}\equiv\boldsymbol{\mu}\cdot\boldsymbol{H}$. Note that $\boldsymbol{E}$ and $\boldsymbol{H}$ are not perpendicular to $\boldsymbol{k}$, but $\boldsymbol{D}$ and $\boldsymbol{B}$ are. To derive a dispersion relation for electromagnetic waves, it is reasonable to treat $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ as constants over one wavelength. One then obtains

$$(\epsilon_\parallel\mu_\parallel\omega^2-k^2)\boldsymbol{B}+\left(1-\frac{\mu_\parallel}{\mu_\perp}\right)[k^2\boldsymbol{n}-\boldsymbol{k}(\boldsymbol{n}\cdot\boldsymbol{k})](\boldsymbol{n}\cdot\boldsymbol{B})+\left(1-\frac{\epsilon_\parallel}{\epsilon_\perp}\right)[\boldsymbol{B}\cdot\boldsymbol{n}\wedge\boldsymbol{k}](\boldsymbol{n}\wedge\boldsymbol{k})=0 \tag{6.5}$$

This leads to two possible speeds for the waves, one for waves with polarisation perpendicular to the direction of anisotropy, $\boldsymbol{n}$, and one for waves polarised in the plane defined by $\boldsymbol{n}$ and $\boldsymbol{k}$. In the first case, $\boldsymbol{n}\cdot\boldsymbol{B}=0$, and the speed is

$$c_\perp=\frac{1}{R}\left(1-(1-\cos\psi)\sin^2\theta\right)^{1/2} \tag{6.6}$$

where θ is the angle between the direction of anisotropy and the direction of propagation of the waves.

On the other hand, when their magnetic field lies in the plane that contains $\boldsymbol{n}$ and $\boldsymbol{k}$, waves propagate at speed

$$c_\parallel=\frac{1}{R}\left(1-\left[(1-\cos\psi)+(2X-1)\sin^2\psi/\cos\psi\right]\sin^2\theta\right)^{1/2} \tag{6.7}$$

There are two contributions to (6.6) and (6.7) which should be carefully distinguished. One arises simply from the fact that the waves propagate on a NGT plane-symmetric background. Indeed, consider a source situated at a distance r from the Earth, with the vector $\hat{\boldsymbol{r}}$ making an angle θ with $\boldsymbol{n}$ (or $\hat{\boldsymbol{x}}$). From (2.1), the equation for light propagation is

$$dt^2-e^\lambda dx^2-\beta(dy^2+dz^2)=0 \tag{6.8}$$

Then, in the (R,ψ) representation, and using $e^\lambda=R^2$ from (4.14), the speed of the light emitted by the source as measured by an observer on Earth would be

$$\frac{dr}{dt}=\frac{1}{R}\left[1-(1-\cos\psi)\sin^2\theta\right]^{1/2} \tag{6.9}$$

which is precisely $c_\perp$. In fact, only the term proportional to $\sin^2\psi$ in $c_\parallel$ is a consequence of the direct coupling between NGT and electromagnetism.

The resulting effect bears a lot of similarity to birefringence in crystals with one optical axis. The polarisation vector of an electromagnetic wave rotates along its trajectory in a way that depends on the difference between the two speeds. Assuming that ψ is very small all along the trajectory in a dust-filled universe ($\gamma=0$), this is:

$$\begin{aligned} c_\parallel-c_\perp &\approx \frac{1}{R}(X-\tfrac{1}{2})\sin^2\psi\sin^2\theta \\ &\approx \frac{1}{R}(X-\tfrac{1}{2})\left(\frac{\sigma}{R_0^3}\right)^2\frac{1}{(6\pi\rho_0)^2}\frac{1}{t^2} \end{aligned} \tag{6.10}$$

The only parameter-free prediction here is that waves propagating along the anisotropic direction will experience no rotation of their polarisation vector.

7. Conclusion

Equations (3.6) give the solution to the vacuum homogeneous, plane-symmetric case in NGT, whereas equations (4.14) represent the equivalent solution in the presence of a perfect fluid. Both exhibit zeros in g_{22} and in $g_{[23]}$. The actual physical meaning of this behaviour is not known at present. In the solution with matter, the NGT parameter σ can be chosen so that the zeros occur early enough in the evolution of the universe for them to be essentially indistinguishable from the initial singularity. Thereafter this anisotropic universe approaches a FRW universe in such a way that currently well-established features of its early history (such as helium production) do not change significantly.

It is interesting that these signature reversals do not occur in a complex Hermitian version of NGT, with $g_{\mu\nu} = g_{(\mu\nu)} + e\, g_{[\mu\nu]}$, where $e^2 = -1$. Taking $\beta = R^2 \cosh\psi$ and $f = R^2 \sinh\psi$, the solution (3.6) now becomes

$$\begin{aligned} e^{\lambda-\lambda_0} &= (t/t_0)^{\frac{-2(1+\kappa^2)}{3-\kappa^2}} \\ R/R_0 &= (t/t_0)^{\frac{2}{3-\kappa^2}} \\ \psi &= \psi_0 + \frac{4\kappa}{3-\kappa^2}\log(t/t_0) \end{aligned} \tag{7.1}$$

with κ a real constant. Therefore, β goes like $t^{\frac{4(1+\kappa)}{3-\kappa^2}}(1 + t^{\frac{-8\kappa}{3-\kappa^2}})$ and does not go through zero. If $\kappa^2 > 1$, it even has a minimum at $t = (\frac{\kappa-1}{\kappa+1})^{\frac{3-\kappa^2}{8\kappa}}$.

In the case with matter, the solution (4.14) becomes

$$\begin{aligned} R &= \left(x^2+\eta^2\right)^{\frac{1}{3(1-\gamma)}} \\ \psi &= \frac{4}{\sqrt{3}}\frac{1}{\gamma-1}\arctan\left(\frac{\eta}{x}\right) \\ e^{\lambda} &= R^2 \\ \rho &= \rho_0 (R/R_0)^{-3(1+\gamma)} \end{aligned} \tag{7.2}$$

where $\eta = \sigma/\sqrt{32\pi b}$ is real, and x is still obtained from (4.13). As $t \to 0$ (or $x \to 0$), $\psi \to (2\pi/\sqrt{3})/(\gamma-1)$, and $\beta \to \eta^{4/3(1-\gamma)}\cosh[(2\pi/\sqrt{3})/(1-\gamma)]$, again without any oscillation through zero.

It has been asserted (Kelly and Mann 1986, Mann and Moffat 1982) that only the hypercomplex structure leads to a linearised NGT that is free of ghost poles. Yet, as mentioned in the introduction, hypercomplex NGT does have ghost poles generically if the $g_{[0i]}$ modes are not somehow decoupled to all orders. Thus, it may be premature to reject the Hermitian structure, especially since, at least in the cases discussed here, the solutions do not exhibit the puzzling signature reversals of the hypercomplex theory. Whether their singularity structure differs from that encountered in GR, as suggested by (7.1) and (7.2), can only be established by a more thorough investigation.

Finally, an intriguing possibility is that some small residual anisotropy may affect the propagation of light in such a way that its polarisation vector rotates as it travels over cosmological distances. Such an effect has been reported recently (Nodland and Ralston 1997a, 1997b, 1997c), but the validity of these observations has been disputed (Carroll and Field 1997, Eisenstein and Dunn 1997, Leahy 1997, Loredo *et al* 1997, Wardle *et al* 1997). If they were confirmed, NGT would provide a possible explanation; if not, an upper bound could be obtained on the size of the coupling of NGT to electromagnetism.

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